ON MAGNETOHYDRODYNAMIC FLOWS OF MIXED TYPE

(O MAGNITOGIDRODINAMICHESKIKH TECHENIIAKH SMESHANNOGO TIPA)

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In hydrodynamics there is only one type of mixed flow, associated with the transition from elliptic subsonic flow to hyperbolic supersonic flow.

As shown below, in magnetohydrodynamics there are several types of mixed flows, described not only by Tricomi's equation $\phi_{yy} - y\phi_{xx} = 0$, but also by the equation $\phi_{xx} - y\phi_{yy} = 0$, which has essentially different properties from the other. Boundary-value problems for these equations are formulated differently than in the case of Tricomi's equation in ordinary gasdynamics, since they have to be satisfied by functions having a different physical significance.

We consider below certain types of mixed flows. For each of them we derive the equations and laws of similitude. Simple examples of mixed flows in nozzles are given.

1. The equations of magnetohydrodynamics for an ideal gas with infinite electrical conductivity have the form

$$\operatorname{div} \rho \mathbf{V} = 0, \ \operatorname{div} \mathbf{H} = 0, \ \operatorname{rot}(\mathbf{H} \times \mathbf{V}) = 0 \tag{1.1}$$

$$(\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{\nabla p}{\rho} = \frac{1}{4\pi\rho} \mathbf{H} \times \operatorname{rot} \mathbf{H} \qquad \frac{p}{\rho^{\varkappa}} = \text{const} \tag{1.2}$$

Here $\mathbf{H}$ and $\mathbf{V}$ are the vectors of the magnetic field and the velocity, p is the pressure, ρ is the density, and κ is the ratio of the specific heats.

When $\mathbf{V}$ is parallel to $\mathbf{H}$ it follows from the first two equations (1.1) that

$$\frac{H}{\rho V} = \frac{H_x}{\rho V_x} = \frac{H_y}{\rho V_y} = \text{const} \tag{1.3}$$

where H_x, H_y, V_x, V_y are the components of the vectors of field and velocity parallel to the Cartesian axes of x and y.

The relations (1.3) enable us to eliminate the magnetic field from (1.2), while Bernoulli's equation [1], which is valid for the flow under consideration, enables us to eliminate p and ρ from the equations.

As a result we obtain

$$(a^2 - V_x^2)\frac{\partial V_x}{\partial x} + (a^2 - V_y^2)\frac{\partial V_y}{\partial y} - V_x V_y\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) = 0 \tag{1.4}$$

$$[a^2(M^2 - N^2) + V_x^2 N^2]\frac{\partial V_x}{\partial y} - [a^2(M^2 - N^2) + V_y^2 N^2]\frac{\partial V_y}{\partial x} +$$

$$+ N^2 V_x V_y\left(\frac{\partial V_y}{\partial y} - \frac{\partial V_x}{\partial x}\right) = 0$$

$$\left(M = \frac{V}{a},\ N = \frac{V^\circ}{a} = \frac{H/\sqrt{4\pi\rho}}{a}\right)$$

where a is the speed of sound and V° is the Alfven speed. Let us direct the x-axis along a streamline. Then Equations (1.4) may be written as

$$(1 - M^2)\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0,\quad [M^2(1 + N^2) - N^2]\frac{\partial V_x}{\partial y} - (M^2 - N^2)\frac{\partial V_y}{\partial x} = 0 \tag{1.5}$$

The equations change their type [1] when one of the coefficients of the derivatives passes through zero. Obviously, this occurs when $M = 1$, $M = N$ and $M = N/\sqrt{(1 + N^2)}$. The character of the transition when $M = 1$ and $M = N$ differs according to whether $N > 1$ or $N < 1$. Interest also attaches to the case when $M = N = 1$.

We give below an analysis of these six cases.

We notice that, according to (1.3)

$$N^2/M^2 = (N^2/M^2)_0(\rho/\rho_0) \tag{1.6}$$

where $(N^2/M^2)_0$ is the limiting value of the ratio N^2/M^2 when $M \to 0$, and ρ_0 is the stagnation density. If $(N^2/M^2)_0 < 1$, then in the whole flow $N < M$, and consequently the only possible transition is the transition with $M = 1$ and $N < 1$. Such a transition is qualitatively similar to the transonic transition in classical gasdynamics, studied in the works of F.I. Frankl, S.V. Falkovich, Guderley and others.

Accordingly, new types of mixed flow can arise only when $(N^2/M^2)_0 > 1$, i.e. for high densities of magnetic energy.

All quantities relating to points where $M = 1$ will be denoted, as usual, by a subscript asterisk. Quantities relating to points at which $M = N/\sqrt{(1+N^2)}$ will be denoted by two subscript asterisks. Quantities relating to points at which $M = N$, i.e. the speed is equal to the Alfven speed: $V = V^\circ$, will be distinguished by a superscript $^\circ$.

2. Flow near the curve $M = M_{**}$. Let $V_x = V_{**}(1 + u)$ and $V_y = V_{**}v$, where u and v are small quantities. Retaining in Equations (1.4) only the terms of the lowest order, we obtain

$$(1 - M_{**}^2)u_x + v_y = 0, \qquad [3 + (\varkappa - 2)M_{**}^2]uu_y + v_x = 0 \tag{2.1}$$

where the subscripts x and y indicate derivatives with respect to the corresponding variable. Substituting the variables

$$u = \frac{1 - M_{**}^2}{3 + (\varkappa - 2)M_{**}^2}\,\xi, \quad v = \frac{(1 - M_{**}^2)^2}{3 + (\varkappa - 2)M_{**}^2}\,\eta \tag{2.2}$$

we reduce the system (2.1) to the canonical form

$$\xi_x + \eta_y = 0, \qquad \xi\xi_y + \eta_x = 0 \tag{2.3}$$

The first of Equations (2.3) permits the introduction of a function ψ such that

$$\xi = \psi_y, \qquad \eta = -\psi_x \tag{2.4}$$

Then from the second of Equations (2.3) we have

$$\psi_y\psi_{yy} - \psi_{xx} = 0 \tag{2.5}$$

Let us introduce the Legendre function

$$\Psi(\xi, \eta) = -\psi(x, y) + \xi y - \eta x \tag{2.6}$$

It is obvious that

$$x = -\Psi_\eta, \qquad y = \Psi_\xi \tag{2.7}$$

Transforming in the second equation (2.3) to the hodograph plane ξ, η, we obtain $y_\xi + \xi x_\eta = 0$, or

$$\Psi_{\xi\xi} - \xi\Psi_{\eta\eta} = 0 \tag{2.8}$$

Equation (2.5) is analogous in form to the equation for the velocity potential in classical transonic flow. Similarly, Equation (2.8) coincides with the equation for the stream function or Legendre potential of transonic flow. It is clear that the solutions of equations of type (2.5) or (2.8) obtained for transonic flows will describe quite different flows in the case under consideration. The boundary-value problems also have a different formulation.

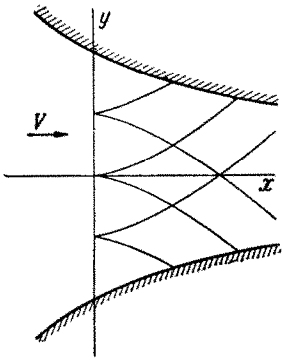

Fig. 1.

The equations of the characteristics in the physical plane and in the hodograph plane have the following forms, respectively:

$$dy = \sqrt{\xi}\, dx, \quad d\eta = \pm\sqrt{\xi}\, d\xi \quad \text{or} \quad \eta = c \pm \tfrac{2}{3}\xi^{3/2} \tag{2.9}$$

The characteristics are real and the equations are of hyperbolic type when $\xi > 0$. Limit lines can arise in the flow only in the hyperbolic region, since the determinant

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = x_\xi y_\eta - x_\eta y_\xi = \xi \Psi_{\eta\eta}{}^2 - \Psi_{\xi\xi}{}^2$$

can vanish only if $\xi > 0$.

Let us consider a simple example of the flow of mixed type under consideration.

It is easy to see that $\psi = cxy$ is a solution of Equation (2.5). According to (2.4) we have $\xi = cx$ and $\eta = -cy$. The streamlines and characteristics of this flow have the respective forms

$$y = c_1/(1+cx), \qquad y = c_2 \pm \frac{2}{3}\sqrt{c}\, x^{3/2}$$

This flow is depicted in Fig. 1. We notice that it has a straight line of transition.

Let us introduce the new variables

$$x = x_0 x^*; \quad y = x_0 \tau^{1/3}\gamma^{1/3}\beta^{-4/3} y^*; \qquad v = \tau v^*, \; u = \tau^{2/3}\beta^{-2/3}\gamma^{-1/3} u^* \tag{2.10}$$
$$(\gamma = 3 + (\varkappa - 2)M_{**}{}^2, \; \beta^2 = 1 - M_{**}{}^2)$$

where x_0 is a characteristic linear dimension of the flow and τ is a characteristic gradient of the velocity vector. It is easy to see that the functions marked with an asterisk satisfy the canonical system (2.3). Two flows, for which the functions u^* and v^* are equal at points with the same values of x^* and y^*, are said to be similar. It is evident that similar flows must have equal values at infinity for the quantity

$$u_\infty^* = \tau^{-2/3}\beta^{2/3}\gamma^{1/3} M_{**}^{-1}\left(1 + \frac{\varkappa - 1}{2} M_{**}{}^2\right)^{-1} \varepsilon = k \tag{2.11}$$

where $\epsilon = M_\infty = M_{**}$ is a small quantity, and M_∞ is the Mach number of the flow at infinity.

The relations (2.10) and (2.11) give the similarity law for the flows under consideration, while the quantity k is the similarity parameter.

The pressure coefficient on a slender body is determined by the expression

$$c_p = \frac{p - p_{**}}{{}^1\!/_2 \rho_{**} v_{**}{}^2} = -2u = -2\tau^{2/3}\beta^{-2/3}\gamma^{-1/3} u^*(k, x^*) \qquad (2.12)$$

If x_0 is the length of the body, then the drag and lift coefficients are given by

$$c_x = \frac{1}{x_0}\oint c_p \tau dx = -2\tau^{5/3}\beta^{-2/3}\gamma^{-1/3} \oint u^*(k;\ x^*) v^* dx^* = -\tau^{5/3}\beta^{-2/3}\gamma^{-1/3} F_1(k)$$

$$c_y = \frac{1}{x_0}\oint c_p dx = -2\tau^{2/3}\beta^{-2/3}\gamma^{-1/3} \oint u^* dx^* = -2\tau^{2/3}\beta^{-2/3}\gamma^{-1/3} F_2(k)$$

3. Trans-Alfven flows. Now let $V_x = V^\circ (1 + u)$ and $V_y = V^\circ v$. Then the simplified system of Equations (1.4) has the form

$$(1 - M^{\circ 2}) u_x + v_y = 0, \qquad u_y^* - u v_x^* = 0 \quad (3.1)$$

When $M^\circ = N^\circ < 1$ the canonical system is

$$\xi_x + \eta_y = 0, \qquad \xi_y - \xi \eta_x = 0$$
$$(\xi = u(1 - M^{\circ 2}),\ \eta = v) \qquad (3.2)$$

Introducing the functions ψ and Ψ by the relations (2.4), (2.6) and (2.7) we obtain

$$\psi_{yy} + \psi_y \psi_{xx} = 0,\ \Psi_{\eta\eta} + \xi \Psi_{\xi\xi} = 0 \quad (3.3)$$

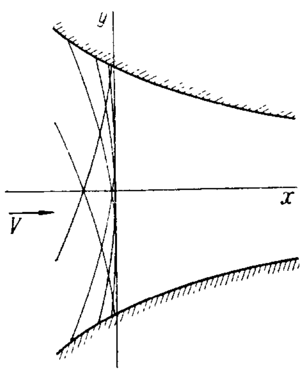

Fig. 2.

Here the equations have the hyperbolic form when $\xi < 0$. Along the characteristics (when $\xi < 0$) we have

$$dy = \pm(-\xi)^{-1/2} dx, \qquad d\eta = \pm(-\xi)^{-1/2} d\xi \qquad \text{or} \quad \eta = c_1 \pm \sqrt{-\xi} \quad (3.4)$$

It is evident that the function $\psi = cxy$ satisfies the first equation (3.3), and consequently $\xi = cx$, $\eta = -cy$. The streamlines of this flow (Fig. 2) coincide with the streamlines of the flow depicted in Fig. 1. The equation of the characteristics is $y = c_1 \pm 2c^{-1/2}\sqrt{-x}$. Here again the transition line is straight.

When $M^\circ = N^\circ > 1$ the canonical system has the form

$$\xi_x - \eta_y = 0, \qquad \xi_y - \xi\eta_x = 0 \tag{3.5}$$

Introducing the functions ψ and Ψ by the relations

$$\xi = \psi_y, \quad \eta = \psi_x, \quad \Psi = -\psi + \xi y + \eta x, \quad x = \Psi_\eta, \quad y = \Psi_\xi \tag{3.6}$$

we obtain

$$\psi_{yy} - \psi_y \psi_{xx} = 0, \qquad \Psi_{\eta\eta} - \xi \Psi_{\xi\xi} = 0 \tag{3.7}$$

These equations are hyperbolic when $\xi > 0$, so that along the characteristics when $\xi > 0$ we have

$$dy = \pm \xi^{-1/2} dx, \quad d\eta = \pm \xi^{-1/2} d\xi \quad \text{or} \quad \eta = c \pm 2\sqrt{\xi} \tag{3.8}$$

The solution $\psi = cxy$ here describes the flow in a conical nozzle (Fig. 3) with streamlines $y = c_1(1 + cx)$ and characteristics $y = c_1 \pm 2c^{-1/2}\sqrt{x}$.

The similarity law for trans-Alfven velocities ($V \sim V^\circ$) is given by the relations

$$x = x_0 x^*, \quad y = x_0 \tau^{-1} y^*, \quad v = \tau v^*, \quad u = \beta^{-2}\tau^2 u^* \tag{3.9}$$

$$k = \tau^{-2}\beta^2 M^{\circ -1}\left(1 + \frac{\varkappa - 1}{2} M^{\circ 2}\right)^{-1} \varepsilon \qquad (\varepsilon = M_\infty - M^\circ) \tag{3.10}$$

where $\beta^2 = 1 - M^{\circ 2}$ when $M^\circ < 1$ and $\beta^2 = M^{\circ 2} - 1$ when $M^\circ > 1$.

The functions with asterisks satisfy the respective canonical systems (3.2) and (3.5). For pressure and force we have

$$c_p = -2\tau^2\beta^{-2} v^*(k, x^*), \qquad c_x = \tau^3\beta^{-2} F_1(k), \qquad c_y = \tau^2 F_2(k)$$

where $c_p = (p - p^\circ)/\ 1/2\ \rho^\circ v^{\circ 2}$. The dependence of the drag on the thickness of the body is found to be stronger than in classical transonic flows.

4. Transonic flows. Now let $V_x = V_*(1 + u)$, $V_y = V_* v$ and $M_* = 1 \neq N_*$. The simplified system (1.4), (1.5) in this case takes the form

$$-(\varkappa + 1) u u_x + v_y = 0, \qquad u_y - (1 - N_*^2) v_x = 0 \tag{4.1}$$

It is easy to see that when $N_* < 1$ Equations (4.1) differ only by the multiplying factor $(1 - N_*^2)$ in the second equation from the system of equations for classical transonic flows. In this case the flow pattern agrees qualitatively with the usual transonic flows.

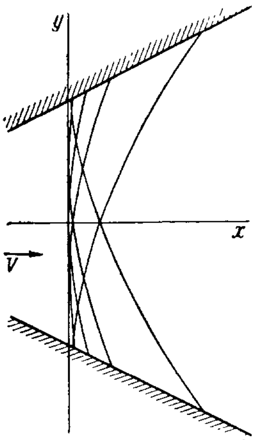

Fig. 3.

When $N_* > 1$ the elliptic and hyperbolic regions change places, i.e. the flow is hyperbolic in the subsonic region ($u < 0$) and elliptic in the supersonic region ($u > 0$). Accordingly, by the substitution $u_1 = -u$ the system (4.1) can be reduced to the ordinary transonic form.

In Fig. 4 we depict such a flow, corresponding to the solution

$$u = cx - \frac{c^2(\varkappa+1)}{2}y^2, \qquad v = c^2(\varkappa+1)xy - \frac{(\varkappa+1)^2c^3}{6}y^3$$

which is analogous to the well-known non-magnetic transonic flow. In Fig. 4 the characteristics are shown by broken curves, while the curve $v = 0$ is shown by a chain-dotted curve.

Of special interest are the flows associated with $M_* = N_* = 1$. In this case the simplified system of equations has the form

$$+(\varkappa+1)uu_x + v_y = 0, \qquad u_y - uv_x = 0 \tag{4.2}$$

Here the flow is of hyperbolic type both when $u < 0$ and when $u > 0$. In the hodograph plane the system (4.2) transforms to

$$-(\varkappa+1)uy_v + x_u = 0, \qquad x_v - uy_u = 0 \tag{4.3}$$

Eliminating the mixed second derivative of x, we obtain

$$u[(\varkappa+1)y_{vv} - y_{uu}] + y_u = 0$$

The characteristics in the hodograph plane and in the physical plane are

$$v = c \pm \sqrt{\varkappa+1}\,u, \qquad dy = \pm(\varkappa+1)^{-1/2}u^{-1}dx \tag{4.4}$$

It is easy to see that the system (4.2) admits a solution of the form

$$u = (x/c)\varphi(y/c), \qquad v = (x/c)\psi(y/c)$$

In particular, we have

$$u = \sqrt{\varkappa+1}\,(x/c)\sec(y/c), \qquad v = (x/c)\tan(y/c)$$

This solution describes the flow depicted in Fig. 5. The flow chart we obtain is symmetric with respect to the straight transition line $x = 0$.

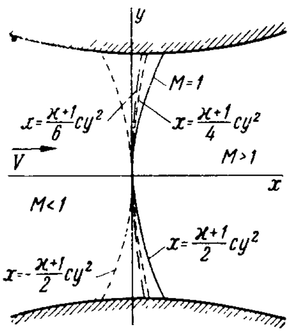

Fig. 4.

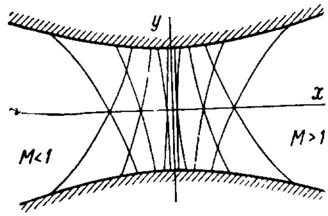

Fig. 5.

Clearly, in such a flow it is completely immaterial whether the gas flows from the subsonic region to the supersonic region or vice versa.

Similar flows are defined by the following relations:

$$x = x_0 x^*, \qquad y = x_0 \tau^{-1} y^*, \qquad u = \tau (1+\varkappa)^{-1/2} u^*, \qquad v = \tau v^* \tag{4.5}$$

$$k = 2\tau^{-1}(1+\varkappa)^{-1/2}\varepsilon \qquad (\varepsilon = M_\infty - 1) \tag{4.6}$$

Correspondingly for pressure and force we have

$$c_p = -2\tau(1+\varkappa)^{-1/2}u^*, \quad c_x = \tau^2 F_1(k), \quad c_y = \tau F_2(k), \quad c_p = \frac{p - p_*}{1/2\,\rho_* v_*^2} \tag{4.7}$$

5. The flows of mixed type analysed above can occur in nozzles in the following combinations:

1) When $(N^2/M^2)_0 < 1$ in a nozzle there is only one region of mixed flow – near the sonic line. This flow is qualitatively similar to the transonic flow of ordinary gasdynamics.

2) When $(N^2/M^2)_0 < \rho_0/\rho_*$, then in addition to the transonic transition of familiar type there are added the transitions when $M = N/\sqrt{(1 + N^2)}$ and $M = N$, which were analysed in Sections 2 and 3, and which bound a subsonic hyperbolic region.

3) When $(N^2/M^2)_0 > \rho_0/\rho_*$ the subsonic hyperbolic region extends from $M = N/\sqrt{(1 + N^2)}$ to the sonic line, where the transition to elliptic supersonic flow analysed in Section 4 takes place. The supersonic elliptic region is brought to an end by the trans-Alfven transition to

hyperbolic flow analysed in Section 3.

4) When $(N^2/M^2)_0 = \rho_0/\rho_*$ the transition from elliptic flow to hyperbolic occurs when $M = N/\sqrt{(1+N^2)} < 1$. The whole flow for large velocities has hyperbolic type with parabolic degeneracy on the sonic line (see Section 4).

BIBLIOGRAPHY

1. Kogan, M.N., Magnitodinamika ploskikh i osesimmetrichnykh techenii gaza s beskonechnoi elektricheskoi provodimost'iu (Magnetodynamic plane and axisymmetric flows of gas with infinite electrical conductivity). *PMM* Vol. 23, No. 1, 1959.

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